Higher-order energy level spacing distributions in the transition region between regularity and chaos

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Abstract. We study general energy level spacing distributions of Hamiltonian systems in the transition region between regularity and chaos. The well known Brody distribution, which results from a power-law ansatz for the level-repulsion function, describes the nearest-neighbour spectral spacings. We pursue an analogous ansatz to determine the level spacing distributions of the kth neighbours, which describe level correlations on longer ranges. The new formula is tested by way of example of the numerical spectra of two different classically chaotic Hamiltonian systems, namely the hydrogen atom in a magnetic field and the Hénon–Heiles system.

1. Introduction

The nearest-neighbour spacing distribution $P(x)$ plays a prominent role in the quantum description of classically chaotic quantum systems (see, e.g. Haake 1991). If the phase space is totally *chaotic*, the distribution of the high-lying energy level spacings is Wigner-like,

$$P(x) = \frac{\pi}{2} x \exp\left(-\frac{\pi}{4} x^2\right). \tag{1}$$

In contrast, in most classically *regular* quantum systems the levels are independent of each other, and therefore the spacings are distributed according to the (zeroth-order) Poisson formula,

$$P(x) = \exp(-x). \tag{2}$$

Brody (1973) proposed a simple one-parameter distribution to describe the level spacings of systems whose phase space dynamics is of a mixed type,

$$P_{\mathrm{bro}}(q, x) = \alpha(q + 1)x^q \exp(-\alpha x^{q+1}) \qquad \alpha = \left[\Gamma\left(\frac{q + 2}{q + 1}\right)\right]^{q+1}. \tag{3}$$

In the special case $q = 0$, the Brody formula (3) turns into the Poisson distribution (regular phase space), and for $q = 1$ the Brody distribution yields the Wigner distribution (chaotic phase space). A value of q between 0 and 1 in many cases gives an indication of the degree of chaoticity of the classical system. In particular, an increase of q may be an indicator of the increase in the ratio of the chaotic phase-space volume to the total kinematically accessible phase-space volume of the corresponding classical system.

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Nevertheless, the semiclassical interpretation of q is still open. Localization effects of chaotic eigenfunctions and the value of the effective Planck constant ($\hbar_{\text{eff}}$) can influence the Brody parameter (Prosen and Robnik 1994).

Although equation (3) still lacks a physical derivation from first principles it can be derived from a power-law ansatz for the level-repulsion function. It is generally accepted that the Brody distribution is a good approximation for high-energy spectra of mixed systems. In Prosen (1995) and Prosen and Robnik (1994) it is argued that the Brody distribution only describes the so-called 'near-semiclassical regime', whereas the so-called 'far-semiclassical regime' of extremely high-lying energy levels is captured by the semiclassical Berry–Robnik distribution (Berry and Robnik 1984). Here we are only interested in the 'near-semiclassical regime', where nonuniform localization of phase-space eigenstates on classically chaotic regions plays an important role (Prosen and Robnik 1994).

In this paper we look for a way of deriving higher-order spacing distributions $P_k(x)$ from a similar Brody-like ansatz. The term $P_k(x)\,\mathrm{d}x$ represents the probability that an interval of length x which starts at a level contains exactly k levels and the next (($k+1$)th) level is in $[x, x+\mathrm{d}x]$. In this notation the nearest-neighbour spacing distribution is denoted by $P_0(x)$. There are two main questions: Is it possible to derive a higher-order spacing distribution from a Brody-like power-law ansatz for the level-repulsion function? And what are the exponents in the different orders of level statistics? For a regular phase space dynamics the levels are totally uncorrelated. Therefore the level spacings are distributed according to the Poisson formula,

$$P_k(x) = \frac{1}{k!} x^k \exp(-x). \tag{4}$$

In section 2 we will derive the general form of higher-order spacing distributions by solving an integral equation for $P_k(x)$, which depends on the level-repulsion function only. As a special case we will propose a spacing distribution which corresponds to a power-law level-repulsion function. In section 3 we will compare our proposed distribution with the numerical data of two Hamiltonian systems: the hydrogen atom in a magnetic field and the Hénon–Heiles system. Both systems are classically chaotic and possess wide transition regions between integrability and chaos. The numerical spectra were obtained by diagonalizing the Hamiltonians at constant scaled energy ε. Under this condition each spectrum corresponds to a specific chaoticity of the underlying classical dynamics.

2. Level spacing distribution

2.1. Nearest-neighbour spacing distribution

We will assume that the nearest-neighbour spacing distribution $P(x) = P_0(x)$ is a normalized probability density with unit expectation value. The latter assumption is necessary for an adequate description of globally homogenous (i.e. unfolded) spectra with unit mean spacing,

$$\int_0^\infty P(x)\,\mathrm{d}x = 1 \qquad \int_0^\infty x\,P(x)\,\mathrm{d}x = \langle x \rangle = 1. \tag{5}$$

Let $F(x)$ be the probability that an interval of length x which starts at a level contains no level. This function results from the corresponding probability density $P(x)$ by integrating over all cases in which the nearest-neighbour level is in the interval $[x, \infty]$. Therefore $F(x)$

is given by

$$F(x) = \int_x^\infty P(y)\,\mathrm{d}y. \tag{6}$$

In this paper the level-repulsion function $r(x)$ plays an important role. It is defined by the ratio

$$r(x) = \frac{P(x)}{F(x)}. \tag{7}$$

The expression $r(x)\,\mathrm{d}x$ is the probability that an interval $[0, x]$ which starts at a level is limited by a level on its right side (i.e. there is a level in $]x, x+\mathrm{d}x[$) *under the condition* that there are no other levels in the interval $]0, x[$ (Bohigas and Giannoni 1984, p 16). Using the language of probability theory (see, e.g. Breiman 1969) we can say that $r(x)\,\mathrm{d}x$ is a conditional probability, i.e. the ratio of the joint probability $P(x)\,\mathrm{d}x$ to $F(x)$. The function $F(x)$ represents the probability of the condition, i.e. the emptiness of the interval $]0, x[$. Because of this probability interpretation the level-repulsion function is sometimes called 'conditional probability density'. It must be stressed, however, that $r(x)$ is *not* a probability density. Rearranging definition (7) we arrive at an integral equation for $P(x)$,

$$P(x) = r(x)F(x) = r(x)\int_x^\infty P(y)\,\mathrm{d}y \tag{8}$$

which can be transformed into a first-order linear differential equation by a simple differentiation,

$$P'(x) = \left[\frac{r'(x)}{r(x)} - r(x)\right] P(x). \tag{9}$$

The formal solution of (9) reads

$$P(x) = r(x)\exp\left[-\int_0^x r(y)\,\mathrm{d}y\right]. \tag{10}$$

Thus, given a special level-repulsion function we can compute the nearest-neighbour spacing distribution from equation (9). The normalized solution of (9) takes the form of the well known Brody distribution if the level-repulsion function is assumed to be of power-law type and the mean spacing is constrained to unity. In explicit terms:

$$r(x) \propto x^q \qquad \text{and} \qquad \langle x \rangle = 1 \Rightarrow P(x) = P_{\mathrm{bro}}(q, x). \tag{11}$$

2.2. Higher-order spacing distributions

In order to study the kth-order spacing distribution, we generalize the equations for the nearest-neighbour statistics ($k = 0$) to neighbours of higher orders ($k \geqslant 1$). The conditions of normalization and unit mean spacing now read

$$\int_0^\infty P_k(x)\,\mathrm{d}x = 1 \qquad \int_0^\infty x P_k(x)\,\mathrm{d}x = \langle x \rangle_k = k + 1. \tag{12}$$

The expressions for the kth-order level-repulsion function $r_k(x)$ and the special function $F_k(x)$ are given by

$$F_k(x) = \int_x^\infty \left[P_k(y) - P_{k-1}(y)\right]\,\mathrm{d}y \tag{13}$$

$$r_k(x) = \frac{P_k(x)}{F_k(x)}. \tag{14}$$

The function $F_k(x)$ represents the probability that an interval of length x, which starts at a level, contains k levels. This probability is computed in equation (13) by integrating over all cases in which the $(k+1)$th level is in the interval $[x, \infty]$. Equation (14) is the kth-order version of equation (7). The conditional probability $r_k(x)\,\mathrm{d}x$ is defined as the probability that an interval $[0, x]$ which starts at a level (at $x = 0$) is limited by a level on its right side (i.e. there is a level in $]x, x + \mathrm{d}x[$) *under the condition* that there are exactly k levels in the interval $]0, x[$. The joint probability $P_k(x)\,\mathrm{d}x$ is defined analogously. To derive the general integral equation for the spacing distribution we note that

$$\int_x^\infty P_k(y)\,\mathrm{d}y = \sum_{j=0}^k F_j(x) = \sum_{j=0}^k \frac{P_j(x)}{r_j(x)}. \tag{15}$$

By differentiation we arrive at an inhomogenous linear differential equation for the kth-order spacing distribution, in which the inhomogenous term includes the solution for the $(k-1)$th order,

$$P_k'(x) = \left[\frac{r_k'(x)}{r_k(x)} - r_k(x) \right] P_k(x) + r_k(x) P_{k-1}(x). \tag{16}$$

The zeroth-order equation is identical to equation (9) for the nearest-neighbour spacing distribution. The system of differential equations is coupled through the inhomogenous term, which includes the solution of the previous order. The general normalized solution of equation (16) can formally be written as

$$R_k(x) := \int_0^x r_k(y)\,\mathrm{d}y \tag{17}$$

$$P_0(x) = r_0(x)\exp[-R_0(x)] \tag{18}$$

$$P_k(x) = r_k(x)\exp[-R_k(x)] \int_0^x P_{k-1}(y)\exp[-R_k(y)]\,\mathrm{d}y. \tag{19}$$

Here, the integral of the level-repulsion function is denoted by $R_k(x)$ for brevity. With the help of equation (19) it is possible to calculate $P_k(x)$, provided that the kth-order level-repulsion function $r_k(x)$ is known. The latter function contains all the information which is necessary to calculate the former distribution. The conditional probability $r_k(x)\,\mathrm{d}x$ corresponds to the joint probability $P_k(x)\,\mathrm{d}x$ as explained before.

   In this paper we propose a (Brody-like) power-law ansatz for the kth-order level-repulsion function,

$$r_k(x) = \rho_k(\rho_k x)^{q_k}. \tag{20}$$

The level-repulsion exponent q_k is a free parameter. It determines the level repulsion of the kth-order spacings. For $k = 0$ it is the parameter $q = q_0$ of the Brody distribution (3). To ensure unit mean spacing, $\langle x \rangle_k = k + 1$, we have to adjust the constants ρ_k. Choosing the power-law form (20) for the level-repulsion function we arrive at a special family of spacing distributions $P_k(x) =: Q_k(x)$, which is a generalization of the Brody distribution (3) to level spacings of higher orders, namely

$$Q_0(x) = \rho_0(\rho_0 x)^{q_0} \exp\left[-\frac{1}{q_0 + 1}(\rho_0 x)^{q_0+1} \right] \tag{21}$$

$$Q_k(x) = \rho_k(\rho_k x)^{q_k} \exp\left[-\frac{1}{q_k + 1}(\rho_k x)^{q_k+1} \right] \int_0^x Q_{k-1}(y)\exp\left[\frac{1}{q_k + 1}(\rho_k y)^{q_k+1} \right]\,\mathrm{d}y. \tag{22}$$

In the following section we will fit the level spacing distributions $Q_k(x)$ to numerical spectra of mixed chaotic systems by adjusting the level repulsion exponents q_k.

3. Comparison with numerical spectra

3.1. Hydrogen atom in a magnetic field

The nonrelativistic hydrogen atom in a magnetic field in cylindrical coordinates (see Hasegawa *et al* 1989 or Friedrich and Wintgen 1989) is described by the Hamiltionan

$$H(p_\rho, p_z, l_z, \rho, z, \gamma) = \frac{1}{2}\left(p_\rho^2 + p_z^2 + \frac{l_z^2}{\rho^2}\right) - \frac{1}{\sqrt{\rho^2 + z^2}} + \frac{1}{8}\gamma^2\rho^2 + \frac{1}{2}\gamma l_z. \tag{23}$$

The Hamiltonian is nonseparable in two degrees of freedom. The remaining symmetries are the rotational symmetry around the direction of the magnetic field (z-axis) and the reflection at the xy-plane. These symmetries can be expressed by the azimuthal quantum number m and the z-parity π.

The field strength parameter γ is physically defined by

$$\gamma = \frac{B}{B_0} \qquad B_0 = \frac{\hbar}{ea_0^2} \qquad a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e}. \tag{24}$$

The Hamiltonian (23) possesses a scaling property. It is invariant under the following transformation:

$$(\tilde{\rho}, \tilde{z}) = \gamma^{\frac{2}{3}}(\rho, z) \qquad (\tilde{p}_\rho, \tilde{p}_z) = \gamma^{-\frac{1}{3}}(p_\rho, p_z) \qquad \tilde{l}_z = \gamma^{\frac{1}{3}}l_z \qquad \varepsilon = \gamma^{-\frac{2}{3}}E. \tag{25}$$

The scaled Hamiltonian $\tilde{H}$ does not include the scaling parameter explicitly, which can formally be expressed by

$$\tilde{H}(\tilde{p}_\rho, \tilde{p}_z, \tilde{l}_z, \tilde{\rho}, \tilde{z}, 1) = \gamma^{-\frac{2}{3}}H(p_\rho, p_z, l_z, \rho, z, \gamma). \tag{26}$$

The classical phase-space dynamics is essentially determined by the choice of the value of the scaled energy ε. By diagonalizing the scaled Hamiltonian at constant scaled energy we obtain a quantum spectrum which corresponds to a fixed type of phase-space dynamics.

For completeness we provide some details on our data analysis: For the Hilbert subspace with $m^\pi = 0^+$ we computed 9749 eigenvalues at $\varepsilon = -0.1$ and 16 223 eigenvalues at $\varepsilon = -0.35$. For $\varepsilon = -0.1$ the classical system is almost completely chaotic. At lower energies some of the trajectories become regular and the phase-space dynamics is of a mixed type, i.e. regular and chaotic trajectories coexist.

From Poincaré surfaces of section we can estimate that for $\varepsilon = -0.35$ a fraction of approximately 15% of the classical phase space is regular (Hasegawa *et al* 1989). We have tested our spacing distributions $Q_k(x)$ for some lower orders $k = 0, \ldots, 7$ at these completely different values of the scaled energy. To separate the local properties of the system from the global ones, we unfold the spectrum. As a second step the lower (nonsemiclassical) part of each spectral series is cut off. To enter the semiclassical region (i.e. $\hbar_{\text{eff}} \to 0$) we disregard the lower 5749 eigenvalues of the spectrum at $\varepsilon = -0.1$ and the lower 6223 eigenvalues of the spectrum at $\varepsilon = -0.35$. For the resulting spectra we compute the neighbour spacing distributions for $k = 0, \ldots, 7$, which should reveal universal characteristics of chaotic quantum systems. To compare the numerical spacing distributions with our theory the free exponents q_k should be adjusted as best-fit parameters. We use a least-squares approximation for the cumulative spacing distribution $\int_0^x Q_k(y)\,dy$. The resulting fits for the three lowest orders are plotted in figure 1. As expected, the nearest-neighbour spacings of the chaotic spectrum ($\varepsilon = -0.1$) are distributed according to the Wigner distribution: $q_0 \approx 1$. For the case of a mixed phase space ($\varepsilon = -0.35$), the zeroth-order distribution reveals the tendency to a more Poisson-like behaviour. The corresponding histogram is a mixture of a Poisson-like and Wigner-like distribution.
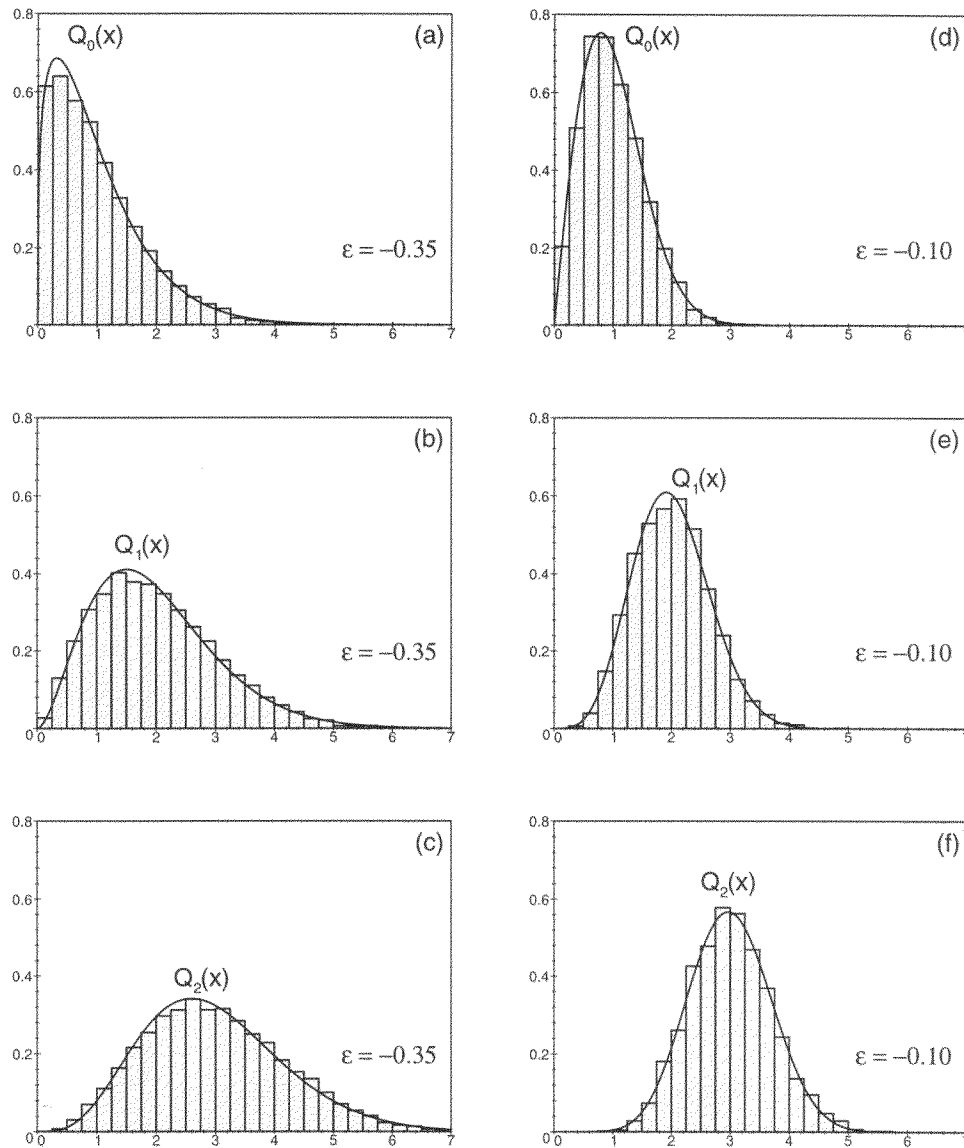
Figure 1. The spacing distributions $Q_k(x)$ of the hydrogen atom in a magnetic field at two different values of the scaled energy ε. The best-fit level repulsion exponents for $\varepsilon = -0.35$ are: (*a*) $q_0 = 0.27$, (*b*) $q_1 = 0.47$, (*c*) $q_2 = 0.63$. For $\varepsilon = -0.1$ the best-fit exponents are: (*d*) $q_0 = 0.97$, (*e*) $q_1 = 1.74$, (*f*) $q_2 = 2.49$. Obviously the histograms are well fitted by the theoretical spacing distributions.

In figure 2 the level repulsion exponents for the lowest eight orders $k = 0, \ldots, 7$ are plotted. Apart from minor deviations the exponents q_k increase linearly with increasing order k. For $\varepsilon = -0.1$ the growth rate is higher than for $\varepsilon = -0.35$. As will be seen below again, this is a typical feature of level repulsion in mixed systems. We assume that the rate of increase of q_k is a measure of spectral chaoticity, i.e. of the chaoticity of the underlying phase-space dynamics. Completely regular spectra are characterized by a vanishing rate of
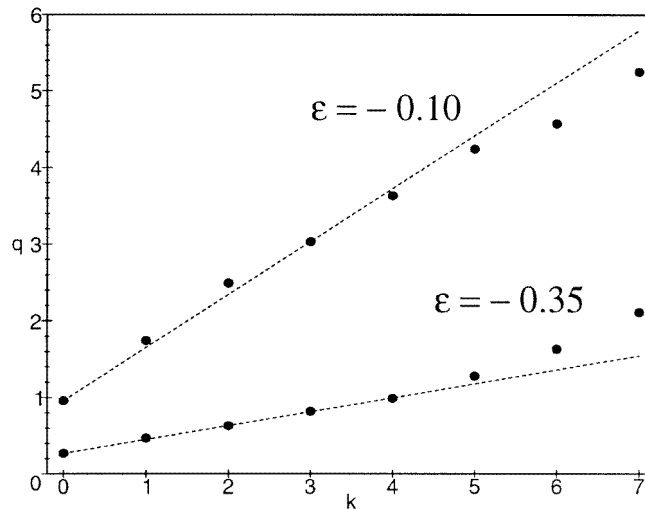
Figure 2. The level repulsion exponents of the hydrogen atom in a magnetic field for two different values of the scaled energy ($\varepsilon = -0.10$ and $\varepsilon = -0.35$). The low-order exponents ($k \leqslant 5$) lie approximately on a straight line. The higher-order exponents ($k > 5$) deviate from this straight line behaviour.

increase. The corresponding Poisson distribution is determined by $q_k = 0$, which means that the level-repulsion functions are constant: $r_k(x) = x^0 = 1$. To check the goodness (in the statistical sense of the word) of our least-squares fits and the reliability of the estimated exponents q_k we perfomed a χ^2 test with 15 degrees of freedom. The energy level spacings were distributed over 16 classes with approximately the same number of spacings in each class. A fit is acceptable at the 5% confidence level if $\chi^2 \leqslant 25$. For $\varepsilon = -0.1$ all fits ($k = 0, \ldots, 7$) are significant. The deviations of q_6 and q_7 are found to be systematic in the following sense: if we choose a different value lying on the straight line, the fits of $Q_6(x)$ and $Q_7(x)$ would become nonsignificant. The goodness of the least-squares fits for $\varepsilon = -0.35$ is slightly worse: the χ^2 values range between 31 and 75. Nevertheless, the histograms are reasonably well fitted by $Q_k(x)$, as can be seen in figure 1. To confirm our assumptions in section 3.2 we will study a completely different Hamiltonian system with mixed chaotic dynamics, namely the Hénon–Heiles system.

3.2. Hénon–Heiles system

This system also possesses a large transition region between regularity and chaos (see, e.g. Lichtenberg and Lieberman 1983). Instead of the original Hamiltonian (Hénon and Heiles 1964) we use a modified form which includes the scaling parameter γ, namely

$$H(p_x, p_y, x, y, \gamma) = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(x^2 + y^2) + \gamma(x^2 y - \tfrac{1}{3}y^3). \qquad (27)$$

The Hénon–Heiles potential has a C_{3v} symmetry. Because of this symmetry the Hilbert space consists of three invariant subspaces A_1, A_2 and E (see, e.g. Jones 1990). The scaling transformations for the modified Hénon–Heiles Hamiltonian read

$$(\tilde{x}, \tilde{y}) = \gamma(x, y) \qquad (\tilde{p}_x, \tilde{p}_y) = \gamma^2(p_x, p_y) \qquad \varepsilon = \gamma^2 E. \qquad (28)$$

It follows that the scaled Hénon–Heiles Hamiltonian $\tilde{H}$ is formally independent of the scaling parameter γ, i.e.

$$\tilde{H}(\tilde{p}_x, \tilde{p}_y, \tilde{x}, \tilde{y}, 1) = \gamma^2 H(p_x, p_y, x, y, \gamma). \qquad (29)$$

The scaled energy determines the degree of chaoticity of the underlying classical dynamics. Increasing the scaled energy from $\varepsilon = \frac{1}{12} \approx 0.108$ up to $\varepsilon = \frac{1}{6} \approx 0.166$ the phase-space undergoes a transition from regularity to chaos. We diagonalize the Hamiltonian (27) at constant scaled energy for a special invariant subspace.

We computed 6062 (5931) eigenvalues in the subspace A_1 (A_2) at $\varepsilon = 0.108$, 4962 (5084) eigenvalues at $\varepsilon = 0.133$ and 2941 (2991) eigenvalues at $\varepsilon = 0.166$. After unfolding the spectra we cut off the low-lying levels: for $\varepsilon = 0.108$ and $\varepsilon = 0.133$ we only consider the highest 4000 eigenvalues of the spectra and for $\varepsilon = 0.166$ only the upper 2000 eigenvalues. From these spectra we computed the cumulative spacing distributions of some orders $k \leqslant 7$ for each value of the scaled energy and for the two subspaces A_1 and A_2. To improve the statistics we mixed the spacings of A_1 and A_2.

From Poincaré surfaces of section the phase-space structure of the Hénon–Heiles system can roughly be estimated (Hénon and Heiles 1964). For $\varepsilon = 0.108 \approx \frac{1}{12}$ the phase space is almost completely regular, for $\varepsilon = 0.133$ nearly a fraction of 50% of the phase space is chaotic and for $\varepsilon = 0.166 \approx \frac{1}{6}$ almost the whole classical phase space is chaotic. The best-fit values of the level repulsion exponents for three different types of classical dynamics $\varepsilon = 0.108$, $\varepsilon = 0.133$ and $\varepsilon = 0.166$ are shown in figure 3. It is seen that the low-order exponents increase linearly. It is a surprising fact that we can reproduce the level spacing distributions of as much as eight orders with a simple power-law ansatz for the level-repulsion function. The good agreement is also evident from figure 4, where histograms and the corresponding best-fit distributions are plotted for comparison. We performed a χ^2 test in the same way as for the hydrogen atom in a magnetic field. Again the fits for the highest scaled energy $\varepsilon = 0.166$ are significant at the 5% confidence level: the χ^2 values range between 6 for $k = 3$ and 24 for $k = 6$. Only the fit for $k = 7$ lies on the
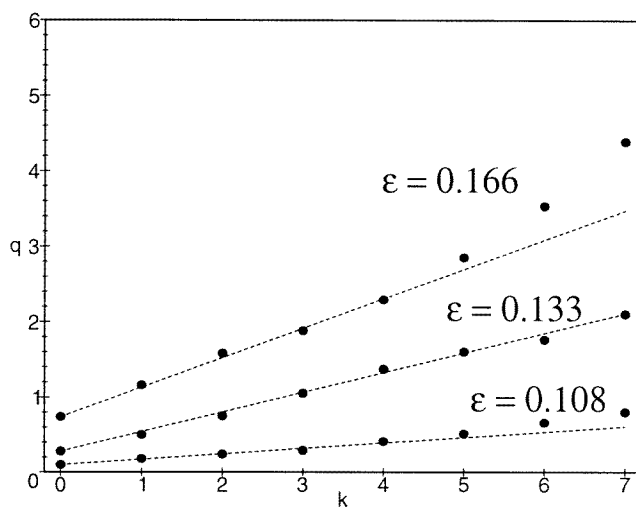

Figure 3. The level repulsion exponents of the Hénon–Heiles system for three different values of the scaled energy ($\varepsilon = 0.108$, $\varepsilon = 0.133$ and $\varepsilon = 0.166$). The low-order exponents (small k) increase linearly wherea s the higher-order exponents show minor deviations froms the straight line behaviour.
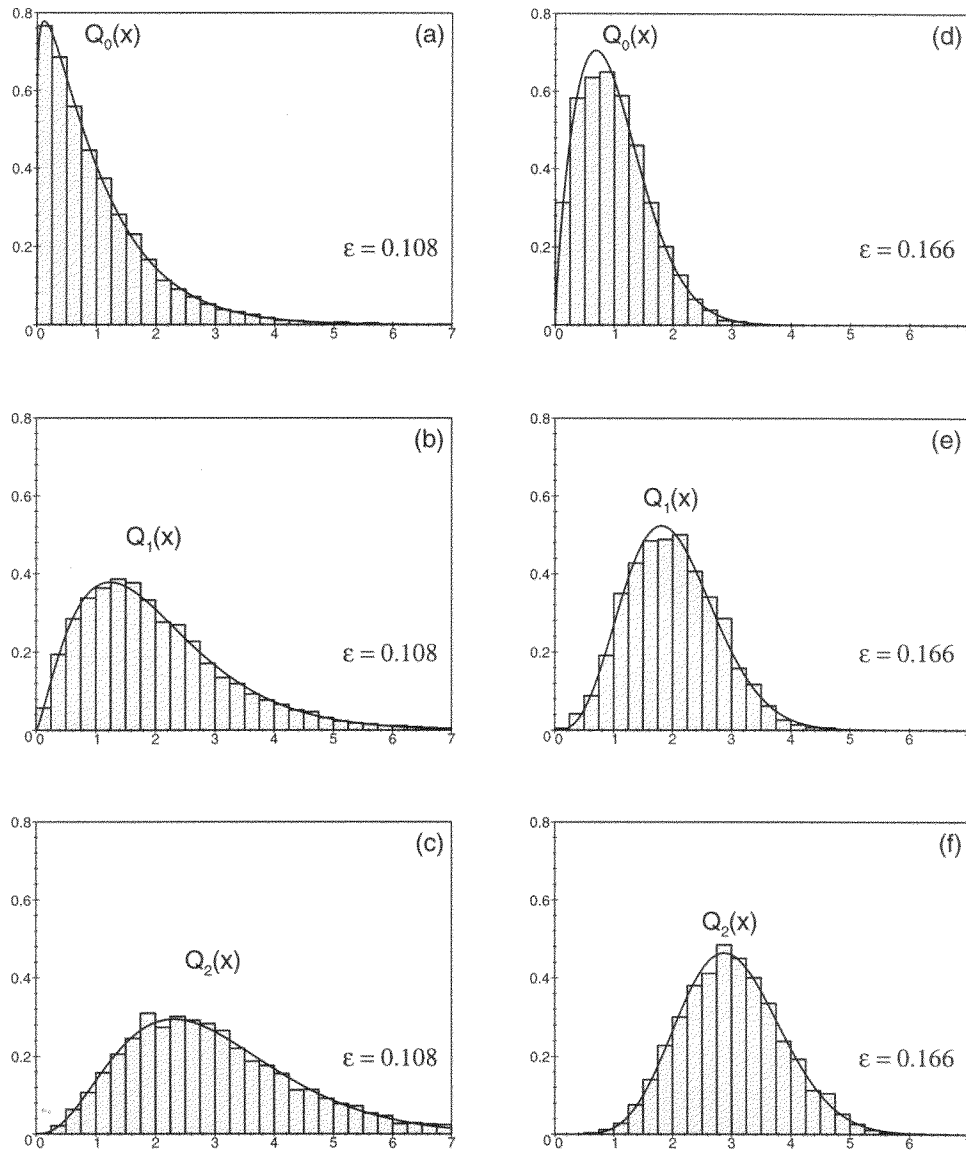
Figure 4. The spacing distributions $Q_k(x)$ of the Hénon–Heiles system at two different values of the scaled energy ε. The best-fit level repulsion exponents for $\varepsilon = 0.108$ are: (*a*) $q_0 = 0.10$, (*b*) $q_1 = 0.18$, (*c*) $q_2 = 0.24$. For $\varepsilon = 0.166$ the best-fit exponents are: (*d*) $q_0 = 0.75$, (*e*) $q_1 = 1.16$, (*f*) $q_2 = 1.58$. There is an excellent agreement between the proposed distributions $Q_k(x)$ and the numerical histograms.

edge of significance: $\chi^2 = 28$. For $\varepsilon = 0.108$ and for $\varepsilon = 0.133$ some higher-order fits are significant and some are not. The lower five orders $k = 0, \ldots, 4$ at $\varepsilon = 0.108$ are significant at the 5% confidence level or at least on the edge of significance.

4. Conclusions

We have generalized the power-law ansatz for the level-repulsion function to higher-order spacing distributions: $r_k(x) \propto x^{q_k}$. As an appropriate interpretation of the level-repulsion function we have shown that $r_k(x)\,dx$ is a conditional probability which corresponds to the joint probability $P_k(x)\,dx$. We formulated a general differential equation for $P_k(x)$ which depends on $r_k(x)$ only. Inserting the power-law ansatz into the solution of the differential equation we arrived at a new type of spacing distributions, $Q_k(x)$. These distributions are an adequate generalization of the Brody distribution to higher orders $k > 0$. The comparison with numerical spectra of two different Hamiltonian systems (the hydrogen atom in a magnetic field and the Hénon–Heiles system) confirmed the validity of the proposed distributions. We also studied the level repulsion exponents q_k at constant scaled energy. In the different orders k of the neighbourhood statistics the exponents of the different orders are correlated to each other. The exponents of the lower orders (small k) approximately lie on a straight line. It is generally accepted that local (short-range) properties of semiclassical quantum spectra reveal universal features of the corresponding phase-space dynamics (see, e.g. Berry 1987). In our model the local properties are represented by level spacing distibutions of lower orders. Therefore we expect that for some k the universal straight line behaviour of the exponents q_k breaks down. Indeed, we found that for $k > 5$ the least-squares fits of $Q_k(x)$ become worse and the best-fit exponents deviate from the straight line behaviour. It is an interesting result of this paper that the short-range spacings ($k \leqslant 5$) of mixed systems effectively depend on one single parameter. The long-range correlations of energy levels are dominated by nonuniversal features which depend on the specific form of the Hamiltonian under consideration. We have shown for two different systems that the energy level distributions $Q_k(x)$ for $k < 7$ are significant at the 5% confidence level if the corresponding classical phase space is chaotic (i.e. $\varepsilon = -0.1$ for the hydrogen atom in a magnetic field and $\varepsilon = 0.166$ for the Hénon–Heiles system). Although a few of the other fits are not significant at the 5% confidence level the corresponding χ^2 values are comparatively small and the overall agreement between the empirical spectra and our distibutions is astonishingly good.

The Brody distribution as well as our distributions $Q_k(x)$ lack a physical derivation from first principles. Prosen and Robnik (1994) derived semiclassically the power-law behaviour of the nearest-neighbour spacing distribution $P_0(x)$ for small spacings $x \ll 1$ in the 'near-semiclassical regime'. However, the derivation of the global Brody-like behaviour is still an open problem. We propose that such a derivation should exploit the simple power-law structure of the level-repulsion function $r_k(x)$.

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